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Error Analysis for Direct Linear Integral Equation Methods*

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Abstract. An error analysis of projection methods for solving linear integral equations of the second kind is presented. The relationships between several direct methods for solving integral equations are examined. It is shown that the error analysis given is applicable to other methods, including a modified Nyström method and certain degenerate kernel methods.

1. Introduction. Consider a linear integral equation of the second kind,

$$\lambda x(s) - \int_a^b k(s, t)x(t)\, dt = y(s), \qquad a \leqq s \leqq b, \tag{1.1}$$

or in operator form

$$(\lambda I - K)x = y. \tag{1.2}$$

The equation is assumed to be in the Banach space $C[a, b]$ of continuous functions on $[a, b]$ normed with the sup norm. We further assume $K: C[a, b] \to C[a, b]$ is a compact operator and that $\lambda \neq 0$ is not an eigenvalue of K. Then the equation has a unique solution $x^*(s)$ for any given $y \in C[a, b]$.

When a projection method is used to find an approximate solution to the above equation, (1.2) is replaced by

$$(\lambda I - P_n K)x_n = P_n y. \tag{1.3}$$

Here P_n is a projection (a linear, idempotent) operator from $C[a, b]$ onto a finite-dimensional subspace S_n of $C[a, b]$. Let M denote a finite-dimensional subspace of the space of continuous linear functionals on $C[a, b]$, and set

$$M_\perp = \{f \in C[a, b] : \mu(f) = 0 \text{ for each } \mu \in M\}.$$

A projection P_n with range S_n and kernel $M_\perp$ is determined if and only if $S_n \cap M_\perp = \{0\}$. If $\{\mu_i\}_1^n$ is a basis of M and $\{y_i\}_1^n$ is a basis of S_n such that

$$\mu_i(y_j) = \delta_{ij}, \qquad i, j = 1, \cdots, n, \tag{1.4}$$

P_n is defined by

$$(P_n f)(s) = \sum_i \mu_i(f) y_i(s), \qquad f \in C[a, b]. \tag{1.5}$$

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When a solution x_n of (1.3) exists, it can be found by choosing a basis $\{\mu_i\}$ of M and a basis $\{u_i\}$ of S_n. Then

$$x_n(s) = \sum_i c_i u_i(s) \tag{1.6}$$

is determined by solving the linear system

$$\sum_j [\lambda\mu_i(u_j) - \mu_i(Ku_j)]c_j = \mu_i(y), \qquad i = 1, \cdots, n, \tag{1.7}$$

for $\{c_i\}$.

Projection methods are examined as a special case of more general approximation methods in [9]. A further analysis with numerical examples is given in [13]. See also [8]. In the next section, we extend the analysis of [13] to include the error due to the use of quadrature in (1.7). A method for constructing quadrature rules for use with projection methods and two examples are given in Section 3. A general class of finite-rank operator methods which includes Nyström and degenerate kernel methods is examined in Section 4, and the analysis of Section 2 is shown to apply to these methods.

2. Error Analysis of Projection Methods. The analysis presented here is motivated by two particular projection methods, collocation and Galerkin's method. The method of collocation is based on projection by interpolation. Thus, the μ_i in (1.5) are given by

$$\mu_i(f) = f(t_i), \qquad t_i \in [a, b], \tag{2.1}$$

and the system (1.7) becomes

$$\sum_j \left[\lambda u_j(t_i) - \int_a^b k(t_i, t)u_j(t)\,dt\right]c_j = y(t_i), \qquad i = 1, \cdots, n. \tag{2.2}$$

Orthogonal or Fourier projection is used in Galerkin's method. If the $\{u_i\}$ satisfy

$$\int_a^b w(t)u_i(t)u_j(t)\,dt = \delta_{ij} \tag{2.3}$$

for some $w(t) \geqq 0$, the functionals in (1.5) are given by

$$\mu_i(f) = \int_a^b w(t)u_i(t)f(t)\,dt. \tag{2.4}$$

In this case, (1.7) becomes

$$\sum_j \left[\lambda\delta_{ij} - \int_a^b w(s)u_i(s)\int_a^b k(s, t)u_j(t)\,dt\,ds\right]c_j = \int_a^b w(s)u_i(s)y(s)\,ds, \qquad i = 1, \cdots, n. \tag{2.5}$$

When the integrals in (2.2) or the inner integrals in (2.5) are replaced by a quadrature rule, the approximating equation being solved is not (1.3), but an equation of the form

$$(\lambda I - P_n K_m)x_{nm} = P_n y. \tag{2.6}$$

Suppose $\{P_n\}$ and $\{K_m\}$ are sequences of operators such that

$$||K - K_m|| \to 0 \quad \text{as } m \to \infty \tag{2.7}$$

and

$$||K - P_n K|| \to 0 \quad \text{as } n \to \infty . \tag{2.8}$$

A technique for constructing operators which satisfy (2.7) is discussed in the next section. Sufficient conditions on K and P_n for (2.8) are given in [13].

THEOREM 2.1. *Conditions* (2.7) *and* (2.8) *imply*

$$\lim_{n\to\infty} \lim_{m\to\infty} ||K - P_n K_m|| = 0. \tag{2.9}$$

Moreover, if $\{||P_n||\}$ *is uniformly bounded,*

$$\lim_{n,m\to\infty} ||K - P_n K_m|| = 0. \tag{2.10}$$

Hence, for all n and m such that

$$||K - P_n K_m||\, ||(\lambda I - K)^{-1}|| < 1, \tag{2.11}$$

a unique solution x_{nm} *of* (2.6) *exists. Furthermore,*

$$||x^* - x_{nm}|| \leqq ||(\lambda I - P_n K_m)^{-1}|| \cdot \{|\lambda|\,(1 + ||P_n||) \operatorname{dist}(x^*; S_n) + ||P_n||\, ||(K - K_m)x^*||\}, \tag{2.12}$$

where $\operatorname{dist}(x^*; S_n) = \inf_{g\in S_n} ||x^* - g||$.

Proof. For each n, (2.7) implies

$$\lim_{m\to\infty} ||P_n K_m - P_n K|| \leqq ||P_n|| \lim_{m\to\infty} ||K - K_m|| = 0.$$

This implies that, for each n, $\lim_{m\to\infty} ||K - P_n K_m|| = ||K - P_n K||$. The result (2.9) follows by letting $n \to \infty$ and employing (2.8). The stronger result (2.10) follows immediately from the relation $||K - P_n K_m|| \leqq ||K - P_n K|| + ||P_n||\, ||K - K_m||$ using (2.7), (2.8), and the uniform boundedness of $\{||P_n||\}$. When (2.11) holds, Banach's theorem [9, p. 172] implies $(\lambda I - P_n K_m)^{-1}$ exists. The bound (2.12) follows from the identity

$$(\lambda I - P_n K_m)(x^* - x_{nm}) = \lambda(x^* - P_n x^*) + P_n(K - K_m)x^*.$$

This completes the proof.

When Galerkin's method is applied, approximations are generally also involved in evaluating the integrals with respect to s in (2.5). Let $\{Q_k\}$ be a sequence of quadrature rules such that, for each $f \in C[a, b]$,

$$Q_k(f) \to \int_a^b w(t)f(t)\, dt \quad \text{as } k \to \infty . \tag{2.13}$$

Define an approximation P_{nk} to the Fourier projection operator P_n by

$$(P_{nk}f)(s) = \sum Q_k(u_i f)u_i(s). \tag{2.14}$$

Then (2.13) implies

$$P_{nk}f \longrightarrow P_n f \quad \text{as } k \longrightarrow \infty \text{, for each } f \in C[a, b]. \tag{2.15}$$

When approximate operators K_m and P_{nk} are used, (1.3) is replaced by

$$(\lambda I - P_{nk}K_m)x = P_{nk}y. \tag{2.16}$$

THEOREM 2.2. *Conditions* (2.7), (2.8) *and* (2.15) *imply*

$$\lim_{n\to\infty} \lim_{k,m\to\infty} ||K - P_{nk}K_m|| = 0. \tag{2.17}$$

For all n, k, m such that

$$||K - P_{nk}K_m|| \, ||(\lambda I - K)^{-1}|| < 1, \tag{2.18}$$

a unique solution $\tilde{x}$ of (2.16) *exists, and*

$$\begin{aligned} ||x^* - \tilde{x}|| \leqq{}& ||(\lambda I - P_{nk}K_m)^{-1}|| \\ & \cdot \{ |\lambda|(1 + ||P_n||) \operatorname{dist}(x^*; S_n) + ||P_{nk}|| \, ||(K - K_m)x^*|| \\ & + |\lambda| \, ||(P_n - P_{nk})x^*|| \}. \end{aligned} \tag{2.19}$$

Proof. We first show that, for each n,

$$\lim_{k,m\to\infty} ||K - P_{nk}K_m|| = ||K - P_nK||. \tag{2.20}$$

Let n be given and fixed. To establish (2.20), it is sufficient to show that

$$\lim_{k,m\to\infty} ||P_nK - P_{nk}K_m|| = 0.$$

The compactness of K and (2.15) imply

$$||P_{nk}K - P_nK|| \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{2.21}$$

(2.15) also implies $\{P_{nk}\}$ is uniformly bounded over k. That is, there exists a number M_n depending only on n, such that $||P_{nk}|| \leqq M_n$ for all k. Now

$$\begin{aligned} ||P_nK - P_{nk}K_m|| &\leqq ||P_nK - P_{nk}K|| + ||P_{nk}K - P_{nk}K_m|| \\ &\leqq ||P_nK - P_{nk}K|| + M_n \, ||K - K_m||. \end{aligned}$$

The result (2.20) now follows using (2.21) and (2.7). Finally, if we take the limit as $n \longrightarrow \infty$ of each side of (2.20) and employ (2.8), we obtain (2.17).

The bound (2.19) is derived from the identity

$$(\lambda I - P_{nk}K_m)(x^* - \tilde{x}) = \lambda\,(x^* - P_nx^*) + P_{nk}(K - K_m)x^* + \lambda(P_n - P_{nk})x^*.$$

Although (2.12) and (2.19) do not generally provide computable error bounds, they are useful in practice for obtaining order of convergence estimates. Examples given in the next section illustrate such usage.

It is interesting to note that the right-hand side y of (1.1) appears explicitly in neither (2.12) nor (2.19). It is the smoothness of $x^* = \lambda^{-1}(y + Kx^*)$, not of y or Kx^* individually, which determines the rate of convergence of the solution of (2.6) or (2.16).

3. Quadrature Rules and Examples. It is not necessary that (2.7) hold in order to successfully use a projection method. However, useful operator approximations $\{K_m\}$ which converge uniformly to K can be easily constructed in many cases.

The approximations given here are similar to those suggested in [1]. In [1], integrals of the form $\int_a^b k(s, t)u(t)\,dt$ are approximated by writing $k_s(t) = k(s, t)$ in the form $k_s(t) = r_s(t)h_s(t)$ where $r_s(t)$ is smooth and $h_s(t)$ can be integrated analytically. The function $r_s(t)u(t)$ is then replaced by an approximation $g_s(t)$ which is of simple form, e.g. a piecewise polynomial. If $h_s(t)$ has been chosen properly, the product $h_s(t)g_s(t)$ can be integrated analytically.

The functions $u_i(t)$ used in (1.7) are generally chosen to have simple form. Hence, we modify the technique above so that only r_s, not $r_s u_i$, is replaced by an approximation. More generally, we have the following:

THEOREM 3.1. *Suppose*

$$k(s, t) = \sum_{p=1}^{q} r_p(s, t)h_p(s, t) \tag{3.1}$$

where, for each p,

$$r_p \in C([a, b] \times [a, b]), \tag{3.2}$$

$$\eta_p = \sup_{a \leq s \leq b} \int_a^b |h_p(s, t)|\,dt < \infty, \tag{3.3}$$

and

$$\sup_{|s-t| \leq \delta} \int_a^b |h_p(s, \tau) - h_p(t, \tau)|\,d\tau \to 0 \quad \text{as } \delta \to 0. \tag{3.4}$$

Let $\{V_m\}$ be a sequence of bounded linear maps from $C[a, b]$ onto the space of bounded integrable functions on $[a, b]$. For each m, define K_m by

$$(K_m f)(s) = \sum_p \int_a^b V_m[r_p(s, t)]h_p(s, t)f(t)\,dt, \qquad f \in C[a, b], \tag{3.5}$$

where it is assumed V_m is applied to r_p as a function of t. Then

$$K_m : C[a, b] \to C[a, b] \tag{3.6}$$

and

$$||K - K_m|| \leqq \alpha_m \sum_p \eta_p \tag{3.7}$$

where

$$\alpha_m = \max_p \sup_{s,t} |r_p(s, t) - V_m[r_p(s, t)]|. \tag{3.8}$$

Hence, if $\alpha_m \to 0$ as $m \to \infty$, then $||K - K_m|| \to 0$.

Proof. Let $\beta = \max_p \sup_{s,t} |r_p(s, t)|$. Fix m, and let $g \in C[a, b]$, and $s, t \in [a, b]$. Then

$$
\begin{aligned}
|(K_m g)(s) - (K_m g)(t)| &\leqq \|g\| \sum_p \int_a^b \{ | V_m[r_p(s, \tau)][h_p(s, \tau) - h_p(t, \tau)]| \\
&\qquad + | V_m[r_p(s, \tau) - r_p(t, \tau)]h_p(t, \tau)| \} \, d\tau \\
&\leqq \|g\| \, \|V_m\| \left\{ \beta \sum_p \int_a^b |h_p(s, \tau) - h_p(t, \tau)| \, d\tau \right. \\
&\qquad \left. + \max_p \sup_\tau |r_p(s, \tau) - r_p(t, \tau)| \sum_p \eta_p \right\}.
\end{aligned}
$$

Thus, (3.6) follows from (3.2)–(3.4). To obtain (3.7), note that, for each $g \in C[a, b]$,

$$\|Kg - K_m g\| = \sup_s \left| \int_a^b \sum_p [r_p(s, t) - V_m(r_p(s, t))]h_p(s, t)g(t) \, dt \right| \leqq \|g\| \, \alpha_m \sum_p \eta_p.$$

This completes the proof.

Collocation Example. Suppose (1.1) is solved approximately by collocation using a cubic spline subspace. Let $\{\pi_n\}$ be a sequence of partitions of $[a, b]$, $\pi_n : a = t_{0n} < t_{1n} < \cdots < t_{nn} = b$ such that $|\pi_n| = \max_i (t_{in} - t_{i-1,n}) \to 0$ as $n \to \infty$. For each n, let S_n denote the subspace of cubic splines with knots on π_n, and let P_n be the interpolation projection onto S_n with interpolating points on π_n and at $\bar{t}_{0n} = (t_{1n} + t_{0n})/2$, $\bar{t}_{nn} = (t_{nn} + t_{n-1,n})/2$. Assume the partitions π_n have uniformly bounded mesh ratios $q_n = |\pi_n|/\min_i (t_{in} - t_{i-1,n})$. Then the projections P_n converge pointwise to the identity [3] and are thus uniformly bounded.

Suppose the kernel function $k(s, t)$ can be expressed in the form (3.1) where, for each p, $r_p \in C^{(2)}[a, b]$ as a function of t, and h_p satisfies (3.3) and (3.4). Let V_m denote the interpolation projection onto the space of linear splines (broken lines) with knots on π_m. Then [4] $\alpha_m = O(|\pi_m|^2)$, so (3.7) implies $\|K - K_m\| = O(|\pi_m|^2)$. Moreover, (2.8) holds as can be seen by applying Theorem 4.1 of [13]. Theorem 2.1 now applies. Thus, a unique solution of (2.6) exists for all sufficiently large m and n. If $x^* \in C^{(k)}[a, b]$, $0 \leqq k \leqq 4$, then [4] dist $(x^*; S_n) = O(|\pi_n|^k)$, so from (2.12), we obtain

$$\|x^* - x_{nm}\| = O(|\pi_n|^k) + O(|\pi_m|^2).$$

Galerkin Example. Let P_n denote the Fourier-Chebyshev projection operator onto the space $\mathbf{P}_n$ of polynomials of degree not greater than n. For each $f \in C[a, b]$, $P_n f$ is given by

$$
\begin{aligned}
(P_n f)(s) &= \sum_{j=0}^{n}{}' I_j(f) T_j(s), \\
I_j(f) &= \frac{2}{\pi} \int_{-1}^{1} (1 - t^2)^{-1/2} T_j(t) f(t) \, dt,
\end{aligned}
\tag{3.9}
$$

where $T_j(s)$ denotes the Chebyshev polynomial of degree j, and $\sum'$ denotes the first term in the summation is to be halved. Suppose $(a, b) = (-1, 1)$ and that (1.1) is to be solved approximately using Galerkin's method with the projection P_n in (3.9).

Using the substitution $t = \cos\theta$, the integrals $I_j(f)$ can be expressed as

$$I_j(f) = \frac{2}{\pi} \int_0^\pi \cos(j\theta) f(\cos\theta) \, d\theta. \tag{3.10}$$

If each I_i is approximated using the trapezoidal rule with spacing $h = \pi/k$, $k \geqq n$, (3.10) is replaced by

$$\tilde{I}_i(f) = \frac{2}{k} \sum_{m=0}^{k}{}'' \cos(j\theta_m) f(\cos\theta_m), \qquad \theta_m = m\pi/k, \tag{3.11}$$

where $\sum''$ denotes the first and last summands are to be halved. The $\tilde{I}_j(f)$ are [7, p. 31] coefficients in the discrete least squares Chebyshev expansion

$$[\tilde{P}_{nk} f](s) = \sum_{j=0}^{n}{}' \tilde{I}_j(f) T_j(s). \tag{3.12}$$

Thus, $\tilde{P}_{nk}$ is itself a projection operator onto $\mathbf{P}_n$. As a consequence, use of the trapezoidal rule to evaluate the integrals involving $w(s) = (1 - s^2)^{-1/2}$ in (2.5) implies that a discrete Galerkin method is actually being used to solve (1.1) approximately. Hence, (2.12) can be used instead of (2.19) to analyze convergence.

When $k = n$, the projection operator $\tilde{P}_{nk}$ becomes interpolation onto $\mathbf{P}_n$ at the points $\cos\theta_m$. In this case [6], $\|\tilde{P}_{nn}\| = O(\ln n)$. Suppose $x^{*(4)}$ exists and is bounded. Then by Jackson's Theorem [11], dist $(x^*; \mathbf{P}_n) = O(n^{-4})$. Thus, (2.12) becomes

$$\|x^* - x_{nm}\| = O(\ln n)[O(n^{-4}) + O(\|(K - K_m)x^*\|)].$$

Instead of (3.11), suppose the Gauss-Chebyshev quadrature formula

$$\int_{-1}^{1} (1 - t^2)^{-1/2} g(t)\, dt = \frac{\pi}{k} \sum_{j=1}^{k} g(\cos\xi_j) + E(g), \qquad \xi_j = (j - \tfrac{1}{2})\pi/k, \tag{3.13}$$

with $k \geqq n + 1$ is used to approximate the integrals in (3.9). The resulting approximation to P_n is given by

$$(P_{nk} f)(s) = \sum_{j=0}^{n}{}' \hat{I}_j(f) T_j(s), \qquad \hat{I}_j(f) = \frac{2}{k} \sum_{m=1}^{k} \cos(j\xi_m) f(\cos\xi_m). \tag{3.14}$$

P_{nk} defines another discrete least squares Chebyshev expansion [7, p. 32]. Thus we are led to a second discrete Galerkin method. Rather than use (2.12) to analyze this method, we illustrate the use of (2.19).

For any function f, the approximation $P_{nk} f$ differs from $P_n f$ by no more than

$$\|P_n f - P_{nk} f\| \leqq \frac{2}{\pi} \sum_{j=0}^{n}{}' |E(T_j f)|\, \|T_j\| = \frac{2}{\pi} \sum_{j}{}' |E(T_j f)|. \tag{3.15}$$

Again assume $x^{*(4)}$ exists and is bounded. Then for each j, dist $(x^* T_j; \mathbf{P}_{n+j}) = O(n^{-4})$, so [5, Section 4.8] $|E(x^* T_j)| = O(n^{-4})$. Thus, (3.15) implies $\|P_n x^* - P_{nk} x^*\| = O(n^{-3})$. A crude bound on $\|P_{nk}\|$ is given by $\|P_{nk}\| \leqq \sum_j' \|\hat{I}_j\|\, \|T_j\| = 2n + 1$, while [11] $\|P_n\| = O(\ln n)$. Thus, (2.19) implies $\|x^* - \tilde{x}\| = O(n^{-3}) + O(n\, \|(K - K_m)x^*\|)$.

4. Finite Rank Operator Methods. Suppose the solution x^* of (1.2) is approximated by the solution $\tilde{x}_n$ of an equation

$$(\lambda I - K_n)\tilde{x}_n = y, \tag{4.1}$$

where K_n is an operator of finite rank. We refer to any such approximation method as a finite rank operator method. In this section, we will show the relation between finite rank operator methods and projection methods. We then apply the analysis of Section 2 to certain of these methods.

Any bounded linear operator K_n of finite rank defined on $C[a, b]$ can be expressed in the form

$$(4.2) \qquad (K_n f)(s) = \sum_{i=1}^{n} \mu_i(f) u_i(s),$$

where the μ_i are bounded linear functionals on $C[a, b]$ and $\{u_i\}$ spans the range of K_n. If the approximating equation (4.1) has a unique solution $\tilde{x}_n(s)$, the solution must satisfy

$$(4.3) \qquad \lambda \tilde{x}_n(s) - \sum_j c_j u_j(s) = y(s), \qquad c_j = \mu_j(\tilde{x}_n).$$

Hence, $\tilde{x}_n(s)$ has the form

$$(4.4) \qquad \tilde{x}_n(s) = \lambda^{-1}\Bigl(y(s) + \sum_j c_j u_j(s)\Bigr).$$

The c_j satisfy the linear system

$$(4.5) \qquad \lambda c_i - \sum_j c_j \mu_i(u_j) = \mu_i(y), \qquad i = 1, \cdots, n,$$

obtained by applying μ_i to each side of (4.3). In fact, $\tilde{x}_n$ given by (4.4) is a solution of (4.1) if and only if the c_j satisfy (4.5). Thus, (4.1) has a unique solution if and only if (4.5) does. The solvability of (4.5) does not depend on whether or not $\{\mu_i\}$ or $\{u_i\}$ is linearly independent.

Now suppose P_n is a projection operator defined by (1.5) and K_n is the operator defined by (4.2), where the functionals μ_i are identical with those in (1.5). If the operator $K_n = K_m$ defined in (4.2) is used in (2.6), the solution x_{nn} of (2.6) is related to the solution of (4.1) by

$$(4.6) \qquad x_{nn} = P_n \tilde{x}_n.$$

To see this, let $\{y_i\}$ be a basis of the range of P_n satisfying (1.4). In solving (2.6), the coefficients d_i in the expansion $x_{nn}(s) = \sum_i d_i y_i(s)$ are determined from the linear system

$$(4.7) \qquad \sum_j [\lambda \delta_{ij} - \mu_i(K_n y_j)] d_j = \mu_i(y), \qquad i = 1, \cdots, n.$$

But (4.2) and (1.4) imply $K_n y_j = \sum_i \mu_i(y_j) u_i = u_j$. Hence, the system (4.7) is identical with (4.5). Since $c_j = \mu_j(\tilde{x}_n)$ and $d_j = \mu_j(x_{nn})$, this implies $\mu_j(\tilde{x}_n) = \mu_j(x_{nn})$, $j = 1, \cdots, n$. Hence, $P_n \tilde{x}_n = P_n x_{nn}$. But $P_n x_{nn} = x_{nn}$, so (4.6) must hold.

The Nyström method and the method of collocation illustrate the relation (4.6). The Nyström method ([12], [2]) is derived by replacing the integral in (1.1) by a quadrature rule

$$(4.8) \qquad \int_a^b g(t)\, dt \doteq \sum_{i=1}^{n} w_i g(t_i).$$

The Nyström method is thus a finite rank operator method where the functionals μ_i in (4.2) are given by point evaluation at t_i and $u_i(s) = w_i k(s, t_i)$. More generally [1], if $k(s, t)$ is expressed in the form (3.1), a product quadrature rule

$$(4.9) \qquad \int_a^b f(t) g(t)\, dt \doteq \sum_{i=1}^{n} w_i g(t_i)$$

might be used rather than (4.8). In this case the $u_i(s)$ have the form $u_i(s) = \sum_p w_{ip}(s)r_p(s, t_i)$.

Suppose the Nyström method is applied to (1.1), and an interpolate $\hat{x}_n = P_n\tilde{x}_n$ of the resulting approximate solution $\tilde{x}_n$ is formed such that $\hat{x}_n(t_i) = \tilde{x}_n(t_i)$, $i = 1, \cdots, n$. Then (4.6) implies that $\hat{x}_n$ is the same function found by solving (1.1) approximately using collocation at the points $\{t_i\}$ with the integrals evaluated using the same quadrature rule (4.8) or (4.9) used in determining $\tilde{x}_n$. This relation between the Nyström method and collocation has been noted before ([9, Section XIV. 4], [14]). The work in Section 2 provides a means of analyzing the error in the approximate solution found.

A second well-known finite rank operator method is the degenerate kernel method. In this method [10], $k(s, t)$ is replaced by a degenerate kernel

$$k_n(s, t) = \sum_{i=1}^{n} \alpha_i(s)\beta_i(t). \tag{4.10}$$

The functionals μ_i in (4.2) are given by

$$\mu_i(f) = \int_a^b f(t)\beta_i(t)\, dt, \tag{4.11}$$

while the u_i are given by $u_i(s) = \alpha_i(s)$, for each i.

One means of obtaining a kernel (4.10) which approximates $k(s, t)$ is to use $k_n(s, t) = P_nk(s, t)$, where P_n is a projection operator applied to $k(s, t)$ as a function of s. The operator K_n in (4.1) now has the form

$$K_n = P_n K. \tag{4.12}$$

The solution $\tilde{x}_n$ of (4.1) satisfies

$$\tilde{x}_n = \lambda^{-1}(y + z_n), \qquad z_n = P_n K\tilde{x}_n, \tag{4.13}$$

where z_n can be found as the solution of

$$(\lambda I - P_n K)z_n = P_n Ky. \tag{4.14}$$

Thus, application of the degenerate kernel method with an approximation operator of the form (4.12) is equivalent to solving the regularized equation [9, p. 552]

$$(\lambda I - K)z = Ky, \tag{4.15}$$

using the method of projections, then defining $\tilde{x}_n$ by (4.13). As we see below, this equivalence permits us to study the error in $\tilde{x}_n$ using the analysis of Section 2.

Note that the solution x^* of (1.2) satisfies $(\lambda I - K)Kx^* = Ky$. Comparing this with (4.15), we see that the solution z^* of (4.15) satisfies $z^* = Kx^*$. Moreover, (4.13) and (4.12) imply $z_n = K_n\tilde{x}_n$. These relations, together with (1.2), (4.1) yield

$$z^* - z_n = Kx^* - K_n\tilde{x}_n = (\lambda x^* - y) - (\lambda\tilde{x}_n - y) = \lambda(x^* - \tilde{x}_n). \tag{4.16}$$

Using (4.14) then (4.15), we have

$$\begin{aligned}(\lambda I - P_n K)(z^* - z_n) &= \lambda z^* - P_n Kz^* - P_n Ky \\ &= \lambda z^* - P_n(\lambda z^* - Ky) - P_n Ky = \lambda(I - P_n)z^*,\end{aligned}$$

so (4.16) implies $(\lambda I - P_nK)(x^* - \tilde{x}_n) = (I - P_n)z^*$. Hence, if S_n = range of P_n, the error in $\tilde{x}_n$ is bounded by

$$(4.17)\qquad ||x^* - x_n|| \leqq ||(\lambda I - P_n K)^{-1}||\,(1 + ||P_n||)\,\mathrm{dist}(z^*;\,S_n).$$

By comparison, we note that if the same projection operator is used in (1.3), the resulting approximation x_n satisfies

$$(\lambda I - P_n K)(x^* - x_n) = \lambda x^* - P_n K x^* - P_n y$$
$$= \lambda x^* - P_n(\lambda x^* - y) - P_n y = \lambda(I - P_n)x^*,$$

so

$$(4.18)\qquad ||x^* - x_n|| \leqq ||(\lambda I - P_n K)^{-1}||\,|\lambda|\,(1 + ||P_n||)\,\mathrm{dist}(x^*;\,S_n).$$

Comparing (4.17) and (4.18), one would expect $\tilde{x}_n$ to be a better approximation to x^* than x_n whenever $z^* = \lambda x^* - y$ can be better approximated than x^* by functions in S_n.

In practice, approximate operators K_m and P_{nk} might be used in place of K and P_n when solving (4.14) numerically. If, instead of (4.14), an approximate equation

$$(4.19)\qquad (\lambda I - P_{nk} K_m)\tilde{z} = P_{nk} K y$$

is solved, the situation is analogous to (2.16) and an error bound for $z^* - \tilde{z}$ can be found using (2.19). Generally, however, one would probably also replace K on the right-hand side of (4.19) by the approximate operator K_m. Thus, the equations

$$(\lambda I - P_{nk} K_m)\tilde{z} = P_{nk} K_m y, \qquad \tilde{x} = \lambda^{-1}(y + \tilde{z})$$

would be solved to obtain an approximation $\tilde{x}$ to x^*. Then

$$(\lambda I - P_{nk} K_m)(x^* - \tilde{x}) = (z - P_n z^*) + (P_n - P_{nk})z^* + P_{nk}(K - K_m)x^*,$$

so analogous to (2.19), we have

$$||x^* - \tilde{x}|| \leqq ||(\lambda I - P_{nk} K_m)^{-1}||$$
$$\cdot\{(1 + ||P_n||)\,\mathrm{dist}(z^*;\,S_n) + ||P_{nk}||\,||(K - K_m)x^*|| + ||(P_n - P_{nk})z^*||\}.$$

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